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Non-local spacetime supersymmetry on the lattice

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Abstract
We show that several well-known one-dimensional quantum systems possess a hidden non-local supersymmetry. The simplest example is the open XXZ spin chain with $\Delta = -1/2$. We use the supersymmetry to place lower bounds on the ground-state energy with various boundary conditions. For an odd number of sites in the periodic chain, and with a particular boundary magnetic field in the open chain, we can derive the ground-state energy exactly. The supersymmetry thus explains why it is possible to solve the Bethe equations for the ground state in these cases. We also show that a similar spacetime supersymmetry holds for the t–J model at its integrable ferromagnetic point, where the spacetime supersymmetry and the Hamiltonian it yields coexist with a global $u(1|2)$ graded Lie algebra symmetry. Possible generalizations to other algebras are discussed.

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1. Introduction

In studying strongly correlated systems, one cannot rely on conventional perturbation theory. It is therefore useful to explore the symmetries of such models in depth. Supersymmetry is a fairly generic term meaning that some of the symmetry generators are fermionic, and so obey anticommutation relations. 'Spacetime' supersymmetry is quite special, because the Hamiltonian not only commutes with the symmetry generators, but is a part of the symmetry algebra—it appears in an anticommutator of fermionic generators. A number of important properties follow from this fact. For example, all the energies obey $E \geqslant 0$.

In this paper we mainly study $\mathcal{N} = 2$ supersymmetry [1]. Here there are two Hermitian-conjugate supercharges, which we denote as Q and $Q^{\dagger}$. The charges are nilpotent, which means they obey $Q^2 = (Q^\dagger)^2 = 0$. Their anticommutation relation yields the Hamiltonian

$$\{Q, Q^\dagger\} = H. \tag{1}$$

An additional bosonic symmetry generator is the fermion number F, which obeys

$$[F, Q] = -Q, \qquad [F, Q^\dagger] = Q^\dagger. \tag{2}$$

Because the Hamiltonian is $QQ^\dagger + Q^\dagger Q$, its eigenvalues E cannot be negative. Any states with $E = 0$ are therefore ground states, and must be annihilated by both Q and $Q^\dagger$. States with $E > 0$ form doublets under the supersymmetry. The two states in the doublet have opposite values of $(-1)^F$, so no states with $E > 0$ contribute to the Witten index $W = \mathrm{tr}(-1)^F\,\mathrm{e}^{-\beta H}$. Thus W is precisely the number of bosonic ground states minus the number of fermionic ground states, independent of β and H.

A number of lattice models result in supersymmetric Lorentz-invariant field theories in the continuum limit [2]. However, only a few lattice models are known where the spacetime supersymmetry defined above is present explicitly on the lattice [3, 4]. The simplest model discussed in [3, 4] consists of spinless fermions with a hard-core interaction, with in particular the restriction that they cannot be on the same or on adjacent sites. It was shown that the energy levels (up to an overall shift) of this theory are the same as that of the XXZ spin chain at a particular value of the anisotropy $\Delta = -1/2$ and with particular twisted boundary conditions. This spin chain here is known to yield a supersymmetric field theory in its continuum limit, so it is not shocking that there is a fermion model in the same universality class which has explicit supersymmetry on the lattice. What is somewhat surprising is that the correspondence between the explicitly-supersymmetric model and the spin chain persists even on the lattice.

This suggests that this XXZ chain is supersymmetric in its own right, so that its Hamiltonian can be written in the form (1). There are no fermions in the XXZ model, so the supercharges must necessarily be non-local combinations of the spins. Such a construction of a fermionic operator from bosonic ones in one dimension is familiar from the Jordan–Wigner transformation of the XXZ model [5].

The purpose of this paper is to show that the XXZ chains at $\Delta = -1/2$ are indeed supersymmetric, and to construct their supercharges. One consequence of this result is that this automatically yields the ground-state energy. The reason is that the supersymmetry requires the Hamiltonian to be of the form $QQ^\dagger + Q^\dagger Q + E_0$, where E_0 is a known (size-dependent) constant. This already means the ground-state energy is bounded from below at E_0, but in some cases we will derive that the ground-state energy is precisely E_0. Analogous results are known as a result of elaborate Bethe ansatz computations [6] and by utilizing the Temperley–Lieb algebra [7], but our result gives this simply and directly. Knowing the ground-state energy exactly in a system solvable by the Bethe ansatz is quite useful, because then the ground state wavefunction can be characterized in terms of the roots of a single polynomial equation [8–10].

Amusingly enough, precisely at this value of Δ, the chain is experimentally realizable by putting a spin-1 chain in a Haldane gap phase in a magnetic field tuned to make one of the spin-triplet excitations degenerate with the ground state [12]. The magnetic field breaks the $SU(2)$ symmetry, and this two-state system becomes an antiferromagnetic XXZ chain at some value of Δ; it is not difficult to show that this is precisely $\Delta = -1/2$. Thus our result provides an experimental realization of supersymmetry!

We also extend our results in several ways. We show that the t–J model at its integrable ferromagnetic point has an explicit supersymmetry, as suggested by the results of [4]. We also present a Hamiltonian which commutes with supercharges R and $R^\dagger$, which obey $R^4 = (R^\dagger)^4 = 0$. This model resembles the XXZ chain, but does not seem to be equivalent.

2. Supersymmetry in the XXZ model

In [4], a series of spacetime supersymmetric lattice models of spinless fermions M_k was constructed. In M_k, the Hilbert space is restricted so that no more than k consecutive sites can be occupied. It was found [4] that for M_1 model, if an edge between two empty sites is mapped to an up-spin, and an occupied site together with its two adjacent edges is mapped to a down-spin, then it is closely related to the XXZ model at a particular coupling. However, the mapping is not always one-to-one, and requires care with the boundary conditions. Thus, although this result strongly suggests the supersymmetry appears directly in the XXZ model, it does not prove it. In this section we show how the supersymmetry indeed appears directly.

The XXZ model is a generalization of the Heisenberg model. The Hamiltonian acts on quantum spins $\vec{S}_j$ in the spin-1/2 representation of $SU(2)$ on each site j:

$$H_{\mathrm{XXZ}} = -h\left(S_1^z + S_L^z\right) - 2\sum_{j=1}^{L-1}\left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z\right) \tag{3}$$

where we have allowed for a boundary magnetic field h. For $\Delta = 1$ and $\Delta = -1$, one recovers the $SU(2)$-symmetric ferromagnetic and antiferromagnetic Heisenberg models respectively. When $|\Delta| \neq 1$, the $SU(2)$ symmetry is broken down to $U(1)$; the conserved charge is, say, the number of down-spins $n_\downarrow$. In this paper we focus mainly on the case $\Delta = -1/2$.

2.1. Open chain

The easiest way to describe the supersymmetry is to utilize a fermionic representation of the spins. We introduce two species of fermions $f_{j\downarrow}$ and $f_{j\uparrow}$, and require that there be one fermion of either species at each site j. Spin operators in the spin-1/2 representation can be written as

$$\vec{S}_j = f_j^{\dagger\alpha}\vec{\sigma}_\alpha^\beta f_{j\beta} \tag{4}$$

where the $\vec{\sigma}$ are Pauli matrices, and α and β are spin indices which can take value $\uparrow$ and $\downarrow$. In terms of raising and lowering operators, we have

$$S_j^+ = f_{j\uparrow}^\dagger f_{j\downarrow}, \qquad S_j^- = f_{j\downarrow}^\dagger f_{j\uparrow}.$$

Using the fermionic representation, we can find supercharges Q and $Q^\dagger$, which commute with the Hamiltonian (3) when $\Delta = -1/2$. To do this, we need to define the 'shift' operator $A_j^{R\dagger}$, which moves all the fermions on sites $k > j$ to the right by one. It thus increases the number of sites in the system by one, while leaving an unoccupied site at $j+1$. The supercharge Q is then given by

$$Q = \sum_{j=1}^{L} Q_j$$

where

$$Q_j = f_{j\uparrow}^\dagger f_{j+1,\uparrow}^\dagger f_{j\downarrow} A_j^{R\dagger}. \tag{5}$$

The supercharge is non-local because of the shift. Simply speaking, the supercharge Q converts a down-spin at site j to two up-spins at j and $j+1$. As a result, the total number of fermions as well as the number of sites is increased by one. To be precise, we define the Hilbert space to have $L+1$ sites with the $(L+1)$th site empty. Then $Q_{L+1} = 0$. The fermion-number generator F in the superalgebra merely corresponds to the generator of $U(1)$ symmetry in the XXZ model:

$$F = \sum_{j=1}^{L} f_{j,\downarrow}^\dagger f_{j,\downarrow}.$$

It is simple to verify that $Q^2 = 0$ by using the fermion anticommutation relations to verify that $Q_j Q_k + Q_{k+1} Q_j = 0$ for $j \leqslant k$ and that $Q_j^2 = 0$. It is instructive to study how these operators change spin configurations. Let the Greek letters $\alpha, \beta, \ldots$ represent up or down-spins, so that the spin configuration $|\alpha\beta \cdots \zeta\rangle$ is the state $f_{1\alpha}^\dagger f_{2\beta}^\dagger \cdots f_{L\zeta}^\dagger|0\rangle$, where $|0\rangle$ is the vacuum state. First, let us check the anticommutator $2Q^2 = \{Q, Q\} = \sum_{i,j}\{Q_i, Q_j\} = 0$. Let us first see how $Q_i Q_j$ for $i < j$ acts. This automatically vanishes except on configurations which have down-spins at the ith and the jth sites. Then Q_j acts on such a configuration $|S\rangle$ as

$$Q_j|S\rangle = Q_j|\ldots\alpha\downarrow\beta\ldots\eta\downarrow\zeta\ldots\rangle = (-1)^{j-1}|\ldots\alpha\downarrow\beta\ldots\eta\uparrow\uparrow\zeta\ldots\rangle$$

where the factor $(-1)^{j-1}$ arises because the three fermionic operators $f_{j,\uparrow}^\dagger f_{j+1,\uparrow}^\dagger f_{j,\downarrow}$ are moved over $j-1$ fermionic sites. Now acting with Q_i gives

$$Q_i Q_j|S\rangle = (-1)^{i+j-2}|\ldots\alpha\uparrow\uparrow\beta\ldots\eta\uparrow\uparrow\zeta\ldots\rangle.$$

If we now compute $Q_{j+1}Q_i$ on the same configuration, we get the same final state, except multiplied by $(-1)^{i+j-1}$. Thus these two contributions to Q^2 cancel. The only terms surviving these cancellations, $Q_{i+1}Q_i$ and Q_i^2, individually vanish. Thus we have proved that Q is nilpotent.

Because $Q^2 = (Q^\dagger)^2 = 0$, a Hamiltonian constructed via $H = \{Q, Q^\dagger\}$ commutes with the charges, and so is supersymmetric. Even though Q increases the number of spins by one, $Q^\dagger$ decreases them by one, so that the Hamiltonian preserves the number of spins. We have

$$H = \sum_{i=1}^{L-1}\sum_{j=1}^{L-1} Q_i Q_j^\dagger + \sum_{j=1}^{L}\sum_{i=1}^{L} Q_j^\dagger Q_i$$

where we use the fact that $Q_{L+1} = 0$ and $Q_L^\dagger = 0$, when acting on our Hilbert space with L fermions and the $(L+1)$th site empty. The only states on which $Q_i Q_j^\dagger$ for $i < j$ are non-vanishing are of the form

$$|S\rangle = |\ldots\alpha\downarrow\beta\ldots\eta\uparrow\uparrow\zeta\ldots\rangle.$$

Acting with $Q_i Q_j^\dagger$ gives

$$Q_i Q_j^\dagger|S\rangle = (-1)^{i+j-2}|\ldots\alpha\uparrow\uparrow\beta\ldots\eta\downarrow\zeta\ldots\rangle.$$

We also have

$$Q_{j+1}^\dagger Q_i|S\rangle = (-1)^{i+j-1}|\ldots\alpha\uparrow\uparrow\beta\ldots\eta\downarrow\zeta\ldots\rangle$$

so that these two terms cancel each other. Similarly, $Q_i Q_j^\dagger$ for $i > j$ cancels with $Q_j^\dagger Q_{i+1}$. Such cancellations get rid of almost all the terms, leaving only

$$H = \sum_{i=1}^{L-1}\left[Q_{i+1}^\dagger Q_i + Q_i^\dagger Q_{i+1} + Q_i Q_i^\dagger\right] + \sum_{i=1}^{L}\left[Q_i^\dagger Q_i\right]. \tag{6}$$

We can now rewrite this Hamiltonian (6) in terms of the spins. The first term acts as

$$Q_{i+1}^\dagger Q_i|\ldots\alpha\downarrow\uparrow\zeta\ldots\rangle = (-1)^{2i-1}|\ldots\alpha\uparrow\downarrow\zeta\ldots\rangle.$$

This means that

$$Q_{i+1}^\dagger Q_i = -S_i^+ S_{i+1}^-.$$

The second term in (6) is likewise $Q_i^\dagger Q_{i+1} = -S_{i+1}^+ S_i^-$. The last term simply counts the number of down-spins: $Q_i^\dagger Q_i = f_{i\downarrow}^\dagger f_{i\downarrow} \equiv n_{i\downarrow}$. Finally, the third counts the number of adjacent up-spins:

$$Q_i Q_i^\dagger = n_{i\uparrow} n_{i+1,\uparrow}.$$

These last two terms can be rewritten in terms of S^z by noting that on any site occupied by a fermion, we have $n_{j\downarrow} + n_{j\uparrow} = 1$, and

$$n_{j\uparrow} = \tfrac{1}{2} + S^z_j.$$

Putting this all together means that the Hamiltonian generated by the supercharge (5) is

$$H = \frac{3}{4}L - \frac{1}{4} - \frac{1}{2}S^z_1 - \frac{1}{2}S^z_L - \sum_{j=1}^{L-1}\left[S^+_j S^-_{j+1} + S^-_j S^+_{j+1} - S^z_j S^z_{j+1}\right]. \tag{7}$$

Comparing with (3) we have

$$H_{\mathrm{XXZ}} = H - \frac{3L-1}{4} \tag{8}$$

where $\Delta = -1/2$ and the boundary magnetic field $h = 1/2$.

The fact that the eigenvalues of H are non-negative means that the spectrum of H_{XXZ} is bounded from below by $-(3L-1)/4$. By using the supersymmetry, we can prove that this is in fact the ground-state energy. The first step is to compute the Witten index $W = \mathrm{tr}(-1)^F\,\mathrm{e}^{-\beta H}$ [1]. As noted in the introduction, under the action of the supersymmetry generators, all states other than the ground state form pairs with the same eigenvalue of H and opposite values of $(-1)^F$. Thus W only receives non-zero contributions from the ground states with zero eigenvalue of H. This means W is independent of β and H: it is just a property of the Hilbert space of states. If W is non-zero, then we know that there is at least one ground state in the XXZ chain with energy precisely $-(3L-1)/4$.

The subtlety in our particular case is that while H preserves the number of sites L, Q effectively increases L by one, while $Q^\dagger$ decreases it. However, note that H_{XXZ} also preserves the number of down-spins $n_\downarrow$, while Q decreases $n_\downarrow$ by 1. Since Q and $Q^\dagger$ preserve the combination $L + n_\downarrow$, it is useful to define the Hilbert spaces $H_{L,n_\downarrow}$. Then Q takes a state in $H_{L,n_\downarrow}$ to $H_{L+1,n_\downarrow-1}$, and $Q^\dagger$ vice-versa. We can then define the Witten index W_N to have the trace taken over all states in the Hilbert spaces with fixed $N \equiv L + n_\downarrow + 1$. A non-zero value means that there is at least one $E = 0$ ground state in the Hilbert spaces with fixed N.

W_N has already been computed for the case at hand, because this reduces to the computation done for an open fermionic chain in [4]. The M_1 model describes spinless fermions hopping on an N-site chain, with a further constraint that no adjacent sites can be simultaneously occupied. Thus there are two states per site (empty and occupied) in the M_1 model, just like in the XXZ model. To match states in the two models with open boundary conditions, we consider configurations in the M_1 model with the first and last site empty. Then each edge between empty sites in M_1 is mapped to an up-spin, while each fermion is mapped to a down-spin [4]. Thus the number of fermions f in M_1 is the number of down-spins $n_\downarrow$, while the length N of the open M_1 chain is related to L by $N = L + n_\downarrow + 1$. The Witten index W_N in the two cases is then identical, but in M_1 it is easy to phrase: it is the sum over all allowed configurations on a chain of length $N-2$ (or equivalently length N with first and last sites unoccupied) with weight $(-1)^F$. This is easily computed using a classical one-dimensional transfer matrix, in the same fashion as one solves the one-dimensional Ising model. One finds that $W_N = 0$ for $N = 3j$, j integer, while $W_N = (-1)^{[N/3]}$ otherwise, where $[N/3]$ denotes the largest integer which is less than $N/3$ [4].

We have thus shown that for $N \neq 3j$, there is a state where H has a zero eigenvalue, so that H_{XXZ} has eigenvalue $-(3L-1)/4$. We now need to translate this into a statement depending on L. This requires finding for a given N, the value of $n_\downarrow$ for the ground state, hence yielding L. Again, this question can be phrased in terms of the open chain discussed in [3]. The key observation is that only ground states are annihilated by both Q and $Q^\dagger$. Assuming n_0

to be the value of $n_\downarrow$ in the ground state, it was shown in [3] that for the closed chain of $N-2$ sites, n_0 is the integer closest to $(N-2)/3$. By applying the decomposition $Q = Q_1 + Q_2$ discussed in [3], it is not difficult to show that this applies to the open chain as well. Basically, one proves that all states annihilated by Q with $n_\downarrow < n_0$ are not annihilated by $Q^\dagger$. Likewise, one proves that all states annihilated by $Q^\dagger$ with $n_\downarrow > n_0$ are not annihilated by Q. For the values of N with non-vanishing W_N, this means that the ground state must have $n_\downarrow = n_0$.

We can now apply these results to the XXZ model (7). For $N = 3n_0 + 1$ for any integer n_0, W_N is non-zero. The integer closest to $3n_0 - 1$ is n_0, so there must be a ground state at $n_\downarrow = n_0$. This corresponds to $L = N - n_\downarrow - 1 = 2n_0$. For $N = 3n_0 + 2$, there is a ground state at $n_\downarrow = n_0$, corresponding to $L = 2n_0 + 1$. Since the boundary magnetic field breaks the $\uparrow\leftrightarrow\downarrow$ symmetry, it does not follow that there is a second ground state for an odd length L. In fact, since the boundary magnetic fields favours up-spins, it is required that the ground state for odd L have more up-spins than down-spins.

To summarize, we have shown that the open XXZ chain with $\Delta = -1/2$ and boundary magnetic field $h = 1/2$ has ground-state energy $-3(L-1)/4$ for any L. For an even number of sites L, the number of down-spins in the ground state is $n_\downarrow = L/2$; for odd L, the ground state has $n_\downarrow = (L-1)/2$. With this knowledge, one should be able to extend the analysis of [8–10] to find the polynomial describing the ground state of this XXZ chain with a boundary magnetic field. We note that our boundary magnetic field is different from the one utilized in [11, 8]. There the field is complex (it arises from demanding quantum-group symmetry), and the Hamiltonian is not Hermitian (although its eigenvalues are real). The XXZ model with our real boundary magnetic field can be solved using the Bethe ansatz [13].

Defining another charge $\widetilde{Q}$ by exchanging all up-spins and down-spins in Q, then due to the obvious symmetry, $\widetilde{Q}^2 = 0$. The Hamiltonian formed by $\widetilde{H} = \{\widetilde{Q}, \widetilde{Q}^\dagger\}$ is therefore also supersymmetric. This yields the same XXZ Hamiltonian except with boundary magnetic field $h = -1/2$, and so has the same ground-state energy $(3L-1)/4$ as well. One can then recover any magnetic field with $|h| \leqslant 1/2$ at the open ends by taking from a linear combination of H and $\widetilde{H}$. This proves that the ground-state energy of the open chain is bounded from below by $-(3L-1)/4$ for any $|h| \leqslant 1/2$, including the case of no magnetic field. However, since Q and $\widetilde{Q}$ do not anticommute, this does not in general result in a supersymmetric Hamiltonian, and the techniques utilized above do not apply. Thus one cannot prove using supersymmetry techniques whether or not the bound is saturated for general h like it is for $h = \pm 1/2$. It is easy to check for $L = 3$ and $L = 4$ that it is not.

2.2. *Periodic chain*

Since the supercharge defined above changes the total number of the spins, it is natural to work on an open spin chain. Nevertheless, by mapping the periodic XXZ chain with $\Delta = -1/2$ to the periodic M_1 chain [4], we can make some general statements about the ground state of the XXZ model. In particular, we derive the ground-state energy when the number of sites is odd.

The correspondence goes as follows [4]. Consider a state $|S\rangle$ in the XXZ model with L sites, and construct an eigenstate of the translation operator T by $|S_t\rangle \equiv |S\rangle + t^{-1}T|S\rangle + \cdots + t^{-(L-1)}T^{L-1}|S\rangle$ for some root of unity t. Then $|S_t\rangle$ is an eigenstate of T with eigenvalue t, provided that

$$t^{-(L-1)}T^L|S\rangle = t|S\rangle. \tag{9}$$

These are twisted boundary conditions when $t^L \neq 1$. Let $|S\rangle$ have $n_\downarrow$ down-spins, and let $m_i \geqslant 1$ be the distance between successive down-spins. Then we can characterize the state $|S_t\rangle$ by a series of $n_\downarrow$ integers $m_1, m_2, \ldots, m_{n_\downarrow}$, with $\sum_{j=1}^{n_\downarrow} m_j = L$. This characterization of the translationally-invariant eigenstates is one-to-one if we identify cyclic permutations of the m_i. Now we do likewise for the model M_1 on N sites. For f fermions one gets a set of integers $l_1, l_2, \ldots, l_f$ with $\sum_{j=1}^{f} l_j = N$. Here the constraint is that $l_j \geqslant 2$, since nearest-neighbour fermions are forbidden in the model M_1. Thus there is a one-to-one map of translationally-invariant eigenstates, if we make the identification $l_i = m_i + 1$, $f = n_\downarrow$, and $N = L + n_\downarrow$. The one catch is that if we demand periodic boundary conditions in M_1, we have $t^N = 1$. This means that we must require twisted boundary conditions in the XXZ model:

$$S^+_{L+j} = t S^+_j, \qquad S^-_{L+j} = t^{-1} S^-_j. \tag{10}$$

Thus $t^N = t^{L+n_\downarrow} = 1$ instead of the usual $t^L = 1$.

With this mapping of states, one can then map the Hamiltonian of the M_1 model to one acting on translationally-invariant eigenstates of the XXZ model. One finds [4]

$$H_{\mathrm{XXZ}}(\text{twisted}) = H_{M_1}(\text{periodic}) - \frac{3L}{4}, \tag{11}$$

where the twisted boundary condition is (10). For periodic boundary conditions in M_1, the Witten index is always non-vanishing, so the lowest eigenvalue of H_{M_1} is zero. This means the lowest eigenvalue in the corresponding sector of the twisted XXZ model is $-3L/4$. Note, however, that different translation eigenvalues correspond to different boundary conditions. Thus when this map is reversed, one finds that XXZ states with periodic boundary conditions can be mapped onto the M_1 model with twisted boundary conditions, again depending on the translation eigenvalue. Unfortunately, twisted boundary conditions in M_1 break the supersymmetry in general, so one can no longer bound the Hamiltonian. The exception is when we look for eigenstates of the XXZ model with periodic boundary conditions in the $t = 1$ sector. These are mapped to the M_1 model with periodic boundary conditions as well.

This means that supersymmetry is present in the periodic XXZ chain only when $t = 1$. Luckily, the ground state of the model is in this sector. This follows from a proof that in a sector with fixed L and $n_\downarrow$, the ground state of the periodic XXZ model is unique [14]. This state must have $t = \pm 1$, because otherwise there would be a second ground state with eigenvalue t^{-1}. With the choice of sign of the $S^x S^x$ and $S^y S^y$ terms in (3), the ground state clearly has $t = 1$. (Besides, for an odd number of sites, the eigenvalue $t = -1$ is not allowed for periodic boundary conditions.) This is not surprising, given that when writing the Hamiltonian acting on eigenstates of T, the entries in the Hamiltonian are the most negative for $t = 1$.

The results of [3, 4] therefore give the ground-state energy of the XXZ model with periodic boundary conditions when L is odd. When N is an not a multiple of 3, the number of fermions in the ground state f_0 is the integer closest to $N/3$, and the translation eigenvalue is $t = 1$. Thus when $N = 3f_0 + 1$, the map to XXZ takes this to $L = N - f_0 = 2f_0 + 1 = 2n_\downarrow + 1$. We thus know that the ground state for an odd number of sites has energy $-3L/4$. When $N = 3f_0 - 1$, the map to XXZ takes this to a state with $L = 2n_\downarrow - 1$. Thus we recover both ground states of the periodic XXZ chain with an odd number of sites.

When N is a multiple of 3, there are two zero-energy eigenstates of M_1, which have $t \neq 1$. This state maps to an XXZ model with an even number of sites, but with twisted boundary conditions. It therefore says nothing about the XXZ model with periodic boundary conditions, but does imply that we can find the ground-state energy with twisted boundary conditions. Thus our results are in harmony with the analysis of [9, 10]. This work shows that the roots of the Bethe equations of the XXZ model at $\Delta = -1/2$ are given in terms of a single polynomial equation, when there are an odd number of sites and periodic boundary conditions, or an even number of sites and twisted boundary conditions. These are the supersymmetric cases!

3. Supersymmetry in the t–J model

Another model, denoted as M_2, was explored in depth in [4]. It is a model of spinless fermions on a chain with the constraint that no more than two consecutive sites can be occupied. We can thus think of this, roughly, as a three-state system: empty sites, lone occupied sites and nearest-neighbour occupied sites. This correspondence was used in [4] to map M_2 to two different familiar three-state models. At one coupling, it can be mapped to the spin-1 generalization of the XXZ chain. By repeating the above analysis to find the exact ground-state energy, one presumably could find the single polynomial equation the corresponding Bethe roots obey.

At a value of the coupling where the Hamiltonian preserves the number of each of these three states, M_2 was mapped to the ferromagnetic t–J model. In particular, if an edge between empty sites in the M_2 model is mapped to an up-spin, a lone occupied site to a down-spin, and an edge between two adjacent occupied sites to a hole, then the model is related to the t–J model in the same fashion as the M_1 is related to the spin-$1/2$ XXZ chain. As with the XXZ model, the spectrum of the two was shown to be the same, when certain twisted boundary conditions are utilized. This therefore hints that like the XXZ model, the supersymmetry can be realized directly in this t–J model at this special point. In this section we show that this is so: there exists a non-local spacetime supersymmetry for the t–J model in an enlarged Hilbert space. As a byproduct, we also explicitly show that the M_2 lattice model has an $SU(2)$ symmetry as well.

The one-dimensional t–J model describes fermions with spin hopping along a chain. Double occupancy is forbidden, so it is convenient to think of this as a three-state system, with an empty site being created by a bosonic operator $b_i^\dagger$. The Hamiltonian is

$$H = -t\sum_{i,\sigma}\left(d_{i\sigma}^\dagger d_{i+1,\sigma} + \text{h.c.}\right) + 2\sum_i n_i - L + J\sum_i\left[S_i^z S_{i+1}^z + \frac{1}{2}\left(S_i^\dagger S_{i+1} + S_{i+1}^\dagger S_i\right) - \frac{1}{4}n_i n_{i+1}\right]. \tag{12}$$

where $f_{i\sigma}$ ($\sigma =\uparrow,\downarrow$) annihilates a fermion, so that the composite operator $d_{i\sigma} = b_i^\dagger f_{i\sigma}$ removes a fermion and creates a boson. The usage of the composite operator ensures that no doubly occupied states will be generated. The n_i are the fermion number operators and the S_i are spin operators:

$$n_{i\sigma} = f_{i\sigma}^\dagger f_{i\sigma}$$
$$S_i^z = \tfrac{1}{2}\left(f_{i\uparrow}^\dagger f_{i\uparrow} - f_{i\downarrow}^\dagger f_{i\downarrow}\right) \qquad S_i^\dagger = f_{i\uparrow}^\dagger f_{i\downarrow} \qquad S_i - f_{i\downarrow}^\dagger f_{i\uparrow}.$$

At $J = \pm 2t$, and the particular chemical potential given in (12), the t–J Hamiltonian has a global $u(1|2)$ symmetry rotating the three states on each site into each other [15]. One can thus think of the $J = 2t$ case as the antiferromagnetic $u(1|2)$ Heisenberg model, and the $J = -2t$ case as the ferromagnetic one. Three of the nine generators of $u(1|2)$ symmetry are fermionic, while the other six are bosonic, so this is a graded Lie algebra. The generators are

$$J_{i,1} = S_i^+ = f_{i\uparrow}^\dagger f_{i\downarrow}, \qquad J_{i,2} = S_i^- = f_{i\downarrow}^\dagger f_{i\uparrow}, \qquad J_{i,3} = S_i^z = \tfrac{1}{2}(n_{i\uparrow} - n_{i\downarrow})$$
$$J_{i,4} = (1 - n_{i,\downarrow})f_{i\uparrow}, \qquad J_{i,5} = J_{i,4}^\dagger, \qquad J_{i,6} = (1 - n_{i,\uparrow})f_{i\downarrow}$$
$$J_{i,7} = J_{i,6}^\dagger, \qquad J_{i,8} = 1 - \tfrac{1}{2}n_i, \qquad J_{i,9} = 1.$$

Because of the fermionic generators, such a symmetry is often called 'supersymmetry' in condensed matter physics. One should keep in mind that it is different from the spacetime supersymmetry used above and to be used below. In both cases the Hamiltonian commutes with the symmetry algebra, but for spacetime supersymmetry, the Hamiltonian (1) is a non-trivial part of the algebra.

We find that in the ferromagnetic case $J = -2t$ that, in addition to the graded Lie algebra, there exists a spacetime supersymmetry whose generators are given by $Q^{(\dagger)} = \sum_i Q_i^{(\dagger)}$ with

$$Q_i = A_i K_i (f_{i+1,\uparrow} d_{i\downarrow} - f_{i+1,\downarrow} d_{i\uparrow}). \tag{13}$$

As in the last section, A_i shifts all the sites j with $j > i + 1$ to the $(j - 1)$th sites, while $K_i = (-1)^{\sum_{j<i} n_h}$ is a 'string', depending on the number of holes to the left of site i. As in a Jordan–Wigner transformation, this string is what makes Q fermionic. Both the shift operator A and the string K make Q act non-locally. Physically, Q_i annihilates a pair of electrons with opposite spins at sites i and $i + 1$, creates a hole at site i, and then shift all the sites to the right of i to the left by one. There are two things worth mentioning about the supercharges.

Let us give a little more detail. We first consider $\{Q, Q\}$. It is easy to see that due to the K_i factors, $A_i K_i f_{i+1,\uparrow} d_{i\downarrow} A_j K_j f_{j+1,\uparrow} d_{j\downarrow}$ is cancelled by $A_j K_j f_{j+1,\uparrow} d_{j\downarrow} A_i K_i f_{i+1,\uparrow} d_{i\downarrow}$. Similar cancellations occur for the other three terms of the anticommutator, so Q and $Q^\dagger$ are nilpotent. For $H = \{Q, Q^\dagger\}$, we have

$$\begin{aligned}\{Q, Q^\dagger\} = \sum_{i,j} &\left\{A_i K_i f_{i+1,\uparrow} d_{i\downarrow}, d^\dagger_{j\downarrow} f^\dagger_{j+1,\uparrow} K_j A_j^\dagger\right\} - \left\{A_i K_i f_{i+1,\uparrow} d_{i\downarrow}, d^\dagger_{j\uparrow} f^\dagger_{j+1,\downarrow} K_j A_j^\dagger\right\} \\ &- \left\{A_i K_i f_{i+1,\downarrow} d_{i\uparrow}, d^\dagger_{j\downarrow} f^\dagger_{j+1,\uparrow} K_j A_j^\dagger\right\} + \left\{A_i K_i f_{i+1,\downarrow} d_{i\uparrow}, d^\dagger_{j\uparrow} f^\dagger_{j+1,\downarrow} K_j A_j^\dagger\right\}.\end{aligned} \tag{14}$$

After the cancellations due to the K_i factors, we have the following terms:

(1) $-A_i K_i f_{i+1,\uparrow} d_{i,\downarrow} d^\dagger_{i+1,\uparrow} f^\dagger_{i+2,\downarrow} K_{i+1} A^\dagger_{i+1}$ and its Hermitian conjugate give us the down-spin hopping term, exchanging up- and down-spins, we get the up-spin hopping term.

(2) $-d^\dagger_{i,\uparrow} f^\dagger_{i+1,\downarrow} K_i A_i^\dagger A_i K_i f_{i+1,\uparrow} d_{i,\downarrow}$ and its Hermitian conjugate gives us neighbouring opposite spins exchange term.

(3) $d^\dagger_{i,\uparrow} f^\dagger_{i+1,\downarrow} K_i A_i^\dagger A_i K_i f_{i+1,\downarrow} d_{i,\uparrow}$ and its Hermitian conjugate count the number of bonds between opposite spins.

(4) $A_i K_i f_{i+1,\uparrow} d_{i,\downarrow} d^\dagger_{i,\uparrow} f^\dagger_{i+1,\downarrow} K_i A_i^\dagger$ counts the number of holes.

These terms yield the Hamiltonian

$$H = -\sum_{i,\sigma} \left(d^\dagger_{i\sigma} d_{i+1,\sigma} + \text{h.c.}\right) - 2\sum_i \left[S_i^z S_{i+1}^z + \frac{1}{2}\left(S_i^\dagger S_{i+1} + \text{h.c.}\right) - \frac{1}{4} n_i n_{i+1}\right] + 2N_h \tag{15}$$

where N_h is the total number of holes. Since $N_h = L - f$, where L is the length and f is the number of fermions, we recover the ferromagnetic t–J Hamiltonian (12) up to a shift L. The ground states of both the M_2 model and this ferromagnetic t–J model are discussed in [4].

We can explore the symmetry structure of the t–J model a little more by considering the commutators of the supercharge and the $SU(2)$ generators, namely, S^+, S^- and S^z. Obviously, Q commutes with S^z, while the commutator of Q and the other two give us $Q^{(1)}$, which is very similar to Q, but it annihilates a pair of electrons with same instead of opposite spins. It is easy to see that the square of $Q^{(1)}$ also vanishes, and $Q^{(1)}$ also commutes with the t–J Hamiltonian, but it does not generate the Hamiltonian.

We have shown that the ferromagnetic t–J model has supersymmetry. It is also interesting to ask if the $SU(2)$ symmetry exists in the M_2 supersymmetric fermion model. The answer is yes: the counterpart of S^+ is the operator which annihilates a single fermion on site i and shifts all the sites $j > i$ to the left by one. Since this operation is fermionic, in order to 'defermionize' it, we need a factor $(-1)^{\sum_{j<i} n_j}$, which depends on the number of fermions to

the left of site i. So we see that the supersymmetry and $SU(2)$ symmetry exists in both the t–J and M_2 models. The supercharge in the M_2 model is local, but it is non-local in the t–J model, while the local $SU(2)$ generators in the t–J model become non-local in the M_2 model.

4. Generalization to $R^4 = 0$

With appropriate boundary conditions, the XXZ model on an open chain is known to have a quantum-group symmetry for all Δ [11]. The simplest non-trivial example is at $\Delta = -1/2$, where the quantum group generators $S^{\pm}$ are nilpotent: $(S^{\pm})^2 = 0$. Given the supersymmetry algebra discussed above, this hardly can be a coincidence. However, the two symmetries are not identical: for example, the boundary conditions required for the quantum-group symmetry result in imaginary boundary magnetic fields; such fields obviously cannot be obtained from a Hermitian Hamiltonian like ours.

The quantum-group symmetry exists at any value of Δ. For example, when $\Delta = -\cos(\pi/s)$ for integer s, the quantum group generators obey $(S^{\pm})^{s-1} = 0$. The similarity of supersymmetry and the quantum-group symmetry at $\Delta = -1/2$ led us to attempt to construct XXZ-type lattice models obeying similar nilpotency relations. A similar idea was pursued in [16], in the context of the particle description of sine-Gordon field theory.

We consider a generalization of our method to a model with $R^4 = 0$. We study a lattice model with same Hilbert space as M_1: spinless fermions c_j forbidden to be adjacent, and we take

$$R_j = P_{j+1} P_{j-1} c_j \exp\left[-\mathrm{i}\pi \sum_{l<j} n_l/2\right] \tag{16}$$

where $P_j = 1 - c_j^{\dagger} c_j$ is a projection operator. A Hamiltonian which commutes with the charge $R = \sum_j R_j$ has been constructed [16]; it is

$$H = [R^{\dagger}, R]^2 - R^{\dagger 2} R^2 - R^2 R^{\dagger 2}. \tag{17}$$

There are two nice equalities satisfied by R:

$$R^4 = 0 \tag{18}$$

$$\{R^{\dagger 2}, [R^{\dagger}, R]\} = 0. \tag{19}$$

They can be easily checked by, again, keeping track of the configuration change. The Hamiltonian includes the following terms, translated into XXZ language by the same arguments as given above. The first one is the next-nearest-neighbour hopping term with the coefficient 1; it interchanges

$$\downarrow\uparrow\uparrow \quad \leftrightarrow \quad \uparrow\uparrow\downarrow.$$

The second term is the nearest neighbour hopping term, which in XXZ language changes

$$\begin{array}{lcll} \downarrow\downarrow\uparrow\downarrow & \to & \downarrow\uparrow\downarrow\downarrow & \text{with magnitude } 2; \\ \uparrow\downarrow\uparrow\uparrow & \to & \uparrow\uparrow\downarrow\uparrow & \text{with magnitude } 2-2i; \\ \uparrow\downarrow\uparrow\downarrow & \to & \uparrow\uparrow\downarrow\downarrow & \text{with magnitude } 2-i; \\ \downarrow\downarrow\uparrow\uparrow & \to & \downarrow\uparrow\downarrow\uparrow & \text{with magnitude } 2-i, \end{array}$$

plus all the Hermitian conjugates. The third term is the potential term for the XXZ model; it counts the number of adjacent up-spins and single down-spins, and also assigns the potentials to the following configurations: $\uparrow\downarrow\uparrow$ with coefficient 2; $\uparrow\downarrow\downarrow$ with coefficient 1; $\downarrow\downarrow\uparrow$ with coefficient 1; and $\downarrow\downarrow\downarrow$ with coefficient 0. All these terms look unsymmetric in order to make it look more symmetric and elegant, we add terms generated by another charge which interchanges down- and up-spins in our first charge and changes i to $-i$. After adding two Hamiltonians together, we get

$$H = \sum_j \left\{ S_j^+ S_{j+2}^- + (4-2i) S_j^+ S_{j+1}^- + \text{h.c.} - \frac{1}{2} S_j^z S_{j+1}^z \right\}. \tag{20}$$

This is a generalized XXZ chain which involves next-nearest-neighbour interactions which exist in the xy plane, but not in the z direction. So if we rotate spins around the z axis by certain angles which depend on the site index, the coefficient of this term can be made purely imaginary, and thus can be taken away by adding another term with i replaced by $-i$. Unfortunately, to get this Hamiltonian we needed to add two different Hamiltonians (as in the open XXZ chain with variable boundary field). But since each charge does not commute with sum of the two Hamiltonian, the symmetry does not seem to persist. Thus we are not sure if the above generalization will shed any light on the analysis of the XXZ model at other values of Δ.

5. Conclusion

In this paper, we explicitly constructed the supercharges which generate spacetime supersymmetries for XXZ and t–J chains. We find that an open XXZ chain at $\Delta = -1/2$ with a particular magnetic field at both ends is supersymmetric. The supercharges change the total number of sites L of the spin chain, while conserving the quantity $L + n_\downarrow$. This enabled us to find that the ground-state energy for any L in the magnetic field ensures supersymmetry, and bounds its value for any smaller field. For periodic boundary conditions, we find the exact ground-state energy for an odd number of sites. For an even number of sites, the supersymmetry only survives for twisted boundary conditions. These particular boundary conditions are precisely the cases where the Bethe equations for the XXZ model ground state can be simplified dramatically [9, 10].

We also showed spacetime supersymmetry and global super Lie algebra $u(1|2)$ coexist in the ferromagnetic t–J model at $2t = -J = 2$. Since the supercharges can be constructed out of the generators of $u(1|2)$, the coexistence of the two symmetries might indicate some intimate relations between them; further work is required to uncover the hidden relation.

A common feature of these supercharges is the non-locality. Non-local symmetries arise in a number of interesting two-dimensional classical and one-dimensional quantum systems as hidden symmetries. A famous example is the Yangian symmetry of the $O(3)$ sigma model [17]. We hope the supersymmetry discussed in this paper will improve our understanding of such models, and provide new insights into other problems.

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